

Second Class Constraints in a Higher-Order Lagrangian Formalism

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Abstract

We consider the description of second-class constraints in a Lagrangian path integral associated with a higher-order Δ -operator. Based on two conjugate higher-order Δ -operators, we also propose a Lagrangian path integral with $Sp(2)$ symmetry, and describe the corresponding system in the presence of second-class constraints.

1. Introduction. Recently, we proposed a Lagrangian path integral formalism based on a Δ -operator that is not restricted to be of second order [1]. The main result was a demonstration that the suggested gauge-fixed functional integral is independent of the chosen gauge-fixing function. This generalizes the field-antifield quantization program [2], taken in its more recent reformulation [3, 4, 5], to the case of gauge symmetries based on higher-order Δ -operators. It also provides a very concise way of understanding the mechanism of gauge-independence, even when restricted to the conventional second-order formalism. Higher order quantum corrections to the conventional second-order operator $\Delta^{(2)}$ [2] are expected to arise from the Hamiltonian formalism when operator-ordering ambiguities are taken into account. A first analysis of this more general framework was carried out in ref. [6], and it has more recently been considered from a different point of view [7, 8]. The relevant mathematical structure can be described by a tower of higher antibrackets [9], and their associated algebra [8].

In this paper we shall describe the corresponding construction of a gauge-fixed Lagrangian path integral in the presence of what appropriately may be called second-class constraints [10] within the antibracket formalism. The possible presence of higher-order terms in the Δ -operator makes it necessary to reconsider this notion, and we therefore first propose a set of conditions the second-class constraints must satisfy. A gauge-fixed Lagrangian path integral in the presence of these constraints is then constructed in such a way as to be independent of the chosen gauge-fixing function. In doing this, we are much helped by a theorem, proven in ref. [6], concerning the relation between the second-order operator $\Delta^{(2)}$ and its most general higher-order local quantum deformations. We finally construct the analogous Lagrangian path integral (both with and without second-class constraints) with extended BRST-anti-BRST symmetry.

2. Constraints of Second Class. Once the conventional field-antifield formalism is formulated in terms of the usual antibracket (A, B) , one can consider the problem of reducing the antisymplectic $2N$ -dimensional manifold down to a physical submanifold of dimension $2(N - K)$ by means of $2K$ second-class constraints $\Theta^\alpha(\Gamma)$ [10]. In the usual case, this proceeds quite analogous to the Poisson-bracket treatment of second-class constraints in the Hamiltonian formalism. The crucial ingredients are the following: For the second-class constraints, the antibracket matrix

$$E^{\alpha\beta} \equiv (\Theta^\alpha, \Theta^\beta) \quad (1)$$

must have an inverse $E_{\alpha\beta}$:

$$E_{\alpha\beta} E^{\beta\gamma} = \delta_\alpha^\gamma. \quad (2)$$

This opens up the possibility of introducing a Dirac antibracket as the projection [10],

$$(A, B)_D \equiv (A, B) - (A, \Theta^\alpha) E_{\alpha\beta} (\Theta^\beta, B), \quad (3)$$

i.e., with the property that the Dirac antibracket vanishes when taken of any function $F = F(\Gamma)$ with any of the constraints:

$$(F, \Theta^\alpha)_D = 0 = (\Theta^\alpha, F)_D. \quad (4)$$

One can also introduce a nilpotent Dirac Δ -operator $\Delta_D^{(2)}$, so that the Dirac antibracket (3) equals the failure of $\Delta_D^{(2)}$ to act as a derivation. This operator commutes* with the second-class constraints:

$$[\Delta_D^{(2)}, \Theta^\alpha] = 0. \quad (5)$$

With these ingredients, the corresponding Lagrangian path integral formulation can be carried out very analogous to the case without second-class constraints [10].

Our aim is to generalize this construction to the case of a higher-order Δ -operator. We do this in the framework of local quantum deformations [6]. This means that we are going to infer the proper

*Here, and in the following, $[\cdot, \cdot]$ equals the graded supercommutator: $[A, B] = AB - (-1)^{\epsilon_A \epsilon_B} BA$.

meaning of second-class constraints in the higher-order formalism from the corresponding second-order formulation. We thus start with a second-order operator $\Delta^{(2)}$. Second-class constraints can now be treated [10] by means of functions $\Theta^{(2)\alpha}(\Gamma)$ and a Dirac Δ -operator $\Delta_D^{(2)}$ satisfying the above conditions. We next apply the deformation theorem to construct, with the help of a transformation $U = U(\Gamma, \partial)$, a general higher-order nilpotent Dirac Δ -operator and the associated second-class constraints:

$$\begin{aligned}\Delta_D &= U^{-1}\Delta_D^{(2)}U = \Delta_D^{(2)} + \mathcal{O}(\hbar) \\ \Theta^\alpha &= U^{-1}\Theta^{(2)\alpha}U = \Theta^{(2)\alpha} + \mathcal{O}(\hbar) .\end{aligned}\tag{6}$$

One unusual consequence of this construction is that the second-class constraints $\Theta^\alpha = \Theta^\alpha(\Gamma, \partial/\partial\Gamma)$ in general are *operators*. Nevertheless, by construction,

$$[\Delta_D, \Theta^\alpha] = 0 \quad , \quad [\Theta^\alpha, \Theta^\beta] = 0 \quad ,\tag{7}$$

the latter of these relations of course trivially being satisfied for the $\Theta^{(2)\alpha}$'s. Because the constraints are now operators, we are led to require that physical functions and operators $F = F(\Gamma, \partial/\partial\Gamma)$ commute with the second-class constraints: $[F, \Theta^\alpha] = 0$.

A particularly interesting subset of deformed theories will consist of those for which the deformation is consistent with the original second-class constraints. In this case $[U, \Theta^{(2)\alpha}] = 0$. The second-class constraints of the higher-order formalism then remain unchanged, and they are in particular still functions. Deformations based on transformations U which depend only on Γ^A and projected differentiations $\overleftarrow{\partial}_B \mathcal{P}^B_A$ with

$$\mathcal{P}^A_B \equiv \delta^A_B - (\Gamma^A, \Theta^\alpha) E_{\alpha\beta} (\Theta^\beta \overleftarrow{\partial}_B)\tag{8}$$

are guaranteed to fall into this class. One could restrict oneself to the set of higher-order Δ -operators that arise from such deformations, in which case the following analysis would be much simplified. But we shall here consider the more general framework in which the second-class constraints are allowed to be operators.

Given solutions $W^{(2)}$ and $X^{(2)}$ of the Master Equations

$$\Delta_D^{(2)} e^{\frac{i}{\hbar} W^{(2)}} = 0 \quad , \quad \Delta_D^{(2)} e^{\frac{i}{\hbar} X^{(2)}} = 0 \quad ,\tag{9}$$

we immediately have the solutions

$$e^{\frac{i}{\hbar} W} = U^{-1} e^{\frac{i}{\hbar} W^{(2)}} \quad , \quad e^{\frac{i}{\hbar} X} = U^{-1} e^{\frac{i}{\hbar} X^{(2)}}\tag{10}$$

to the higher-order Master Equations

$$\Delta_D e^{\frac{i}{\hbar} W} = 0 \quad , \quad \Delta_D e^{\frac{i}{\hbar} X} = 0 \quad .\tag{11}$$

These, in turn, can be expanded [7, 8] in terms of the higher antibrackets $\Phi_{\Delta_D}^k$, *viz.*,

$$e^{-\frac{i}{\hbar} W} \Delta_D e^{\frac{i}{\hbar} W} = \sum_{k=0}^{\infty} \left(\frac{i}{\hbar}\right)^k \Phi_{\Delta_D}^k(W, \dots, W) \quad .\tag{12}$$

We can define the k th Dirac antibracket in term of k nested commutators acting on unity [9, 8]:

$$\Phi_{\Delta_D}^k(A_1, \dots, A_k) \equiv [[\dots [\Delta_D, A_1], \dots, A_k] \, 1 \quad .\tag{13}$$

With this definition we can even consider antibrackets of operators. If these operators (including functions) all commute, then all standard properties of these higher antibrackets (such as their strongly

homotopy Lie algebra) are preserved [8]. In particular, we can consider the Dirac antibrackets with one (or more) of the second-class constraints inserted, all other slots being occupied by physical (see above) and mutually commuting operators. By virtue of eq. (7), all these higher Dirac antibrackets vanish identically:

$$\Phi_{\Delta_D}^k(A_1, \dots, A_{i-1}, \Theta^\alpha, A_{i+1}, \dots, A_k) = 0. \quad (14)$$

This is the obvious generalization to higher order of the Dirac antibracket property (4). It means that in the algebra of higher antibrackets based on Δ_D , the Θ^α 's are zero operators in the strong sense.

Consider now the appropriate prescription for the Lagrangian path integral in the presence of these second-class constraints. In the second-order formalism, a solution to this problem was given in ref. [10]. In the general higher-order formalism, the functional measure is such that the Dirac operator Δ_D is symmetric,

$$\Delta_D = \Delta_D^T. \quad (15)$$

Here transposition is defined, for an arbitrary operator \mathcal{A} , by

$$\int d\mu F(\rho_\Theta \mathcal{A} G) = (-1)^{\epsilon_F \epsilon_A} \int d\mu (\mathcal{A}^T F)(\rho_\Theta G), \quad (16)$$

where, because ρ_Θ in general will be operator valued, its position in the integrand is fixed. Here $d\mu = d\Gamma d\lambda \rho(\Gamma)$ is an overall measure, without any reference to the second-class constraints. In the case of a second-order Δ -operator, we combine $\rho(\Gamma)$ and $\rho_\Theta^{(2)}(\Gamma)$ into the antisymplectic Dirac measure $\rho_D(\Gamma) = \rho(\Gamma) \rho_\Theta^{(2)}(\Gamma)$. Note that in the present context

$$\rho_\Theta = U^{-1} \rho_\Theta^{(2)} U = \rho_\Theta^{(2)} + \mathcal{O}(\hbar), \quad (17)$$

where $\rho_\Theta^{(2)}$ corresponds to the second-order formalism with 2nd class constraints $\Theta^{(2)\alpha}$. Terms proportional to \hbar and higher will in general be operator valued.

In the second-order case [10], where indeed the Δ_D -operator is symmetric according to the above definition, the gauge-fixed path integral can be represented in the form

$$Z_X^D = \int d\Gamma d\lambda \rho_D(\Gamma) \prod_\alpha \delta(\Theta^{(2)\alpha}) e^{\frac{i}{\hbar}(W^{(2)} + X^{(2)})}. \quad (18)$$

Remarkably, a closed-form expression for the gauge fixing X can be found in this second-order case[†]. This is the generalization of the expression given in ref. [4] (see also the discussion in ref. [11]). To find it, we first introduce gauge-fixing functions G_a in involution with respect to the Dirac antibracket,

$$(G_a, G_b)_D = G_c U_{ab}^c. \quad (19)$$

Now consider a solution to the Master Equation for $X^{(2)}$ (9) of the form $X^{(2)} = G_a \lambda^a + i\hbar H + \dots$, where the omitted terms are at least proportional to λ^* . Requiring the Master Equation (9) for X to be satisfied, one finds the explicit solution

$$\exp[-H] = \sqrt{\text{sdet}(F^a, G_b)_D J_D \rho_D^{-1}}, \quad (20)$$

where $J_D = \text{sdet}(\bar{\Gamma}^A \overleftarrow{\partial}_B)$ is the Jacobian of the change of coordinates

$$\Gamma^A \longrightarrow \bar{\Gamma}^A \equiv \{F^a; G_a; \Theta^{(2)\alpha}\}. \quad (21)$$

The solution H as defined in eq. (20) is independent of the choice of F^a on the surface $G_a = 0$.

[†]Up to terms proportional to λ^* , which are not of our concern here. See, *e.g.*, ref. [5].

Comparing with the second-order case [10], we now propose the following gauge-fixed path integral in the general case[‡]:

$$Z_X^D = \int d\mu e^{\frac{i}{\hbar}W} \prod_{\alpha} \delta(\Theta^{\alpha}) \rho_{\Theta} e^{\frac{i}{\hbar}X} . \quad (22)$$

Here both W and X satisfy the appropriate higher-order Master Equations (11). The transformation U should be orthogonal. Using the technique of ref. [1], and the condition (5), one verifies the crucial property that this path integral is independent of the gauge fixing X .

The integral (22) is invariant under changes of bases of the second-class constraints Θ^{α} . All our defining equations, including that of the Dirac antibracket itself, have been formulated in the strong sense. Although these equations are unstable under reparametrizations of the constraints, there is no need to consider the corresponding set of weak equations, due to the presence of the $\delta(\Theta^{\alpha})$ -factor in the integrand of (22). This aspect is completely analogous to the situation in the Hamiltonian path integral with second-class constraints.

Despite its apparently asymmetric formulation in eq. (22) we also observe that

$$Z_X^D = \int d\mu e^{\frac{i}{\hbar}W} \prod_{\alpha} \delta(\Theta^{\alpha}) \rho_{\Theta} e^{\frac{i}{\hbar}X} = \int d\mu e^{\frac{i}{\hbar}X} \prod_{\alpha} \delta(\Theta^{\alpha}) \rho_{\Theta} e^{\frac{i}{\hbar}W} , \quad (23)$$

as a result of $(\Theta^{\alpha})^T = \Theta^{\alpha}$ and $\rho_{\Theta}^T = \rho_{\Theta}$. Thus, as in the formulation without second-class constraints, only boundary conditions distinguish the “action” W from its “gauge fixing” X .

3. The $Sp(2)$ Construction. All of the above considerations can be generalized to the case of triplectic quantization, which is based on an $Sp(2)$ -symmetric formulation that includes both BRST and anti-BRST symmetries [12, 5, 13, 14]. Consider first the case where there are no second-class constraints. We need to provide the appropriate generalization of triplectic quantization to the case of higher-order Δ -operators. Fortunately, this generalization can rather easily be inferred from our earlier paper [1]. Introduce a pair of Δ -operators

$$\Delta_{\pm}^a \equiv \Delta^a \pm \frac{i}{\hbar} V^a , \quad (24)$$

with the following algebra (curly brackets indicate symmetrization in the indices):

$$\Delta_{\pm}^{\{a} \Delta_{\pm}^{b\}} = 0 . \quad (25)$$

The operators Δ^a are of second order or higher, while the operator V^a is of first order only. Note also the split in terms of powers of \hbar . From the point of view of quantum deformations of the usual second-order triplectic formalism, the Δ^a -operators of eq. (24) contain infinite expansions in \hbar , starting with the classical terms.

Let there now be given a functional measure $d\mu \equiv d\Gamma d\lambda \rho(\Gamma)$ on a $6N$ -dimensional triplectic space of fields. The operators Δ_{+}^a and Δ_{-}^a are required to be conjugate to each other in the sense that

$$\int d\mu F \Delta_{\pm}^a G = (-1)^{\epsilon_F} \int d\mu (\Delta_{\mp}^a F) G . \quad (26)$$

We define the Master Equations for W and X to be

$$\Delta_{+}^a e^{\frac{i}{\hbar}W} = 0 , \quad \Delta_{-}^a e^{\frac{i}{\hbar}X} = 0 , \quad (27)$$

and then construct on the basis of their solutions the path integral

$$Z_X = \int d\mu e^{\frac{i}{\hbar}[W+X]} . \quad (28)$$

[‡] The δ -function of an operator is given meaning in terms of a suitable representation and its formal Taylor expansion.

The description of the Master Equations (27) in terms of higher antibrackets, and the $Sp(2)$ generalization of the algebra these higher antibrackets satisfy are described in ref. [8].

Consider now operators Δ_{\pm}^a and solutions X to the Master Equation (27) belonging to the class for which

$$e^{\frac{i}{\hbar}X'} = e^{\varepsilon_{ab}\frac{\hbar}{i}[\Delta_{-}^a, [\Delta_{-}^b, \Phi]]} e^{\frac{i}{\hbar}X} \quad (29)$$

is a *maximal* deformation of X which preserves (27) [15, 5]. The partition function (28) is unchanged under this maximal deformation:

$$\begin{aligned} \delta Z_X &= \left(\frac{\hbar}{i}\right) \int d\mu e^{\frac{i}{\hbar}W} \varepsilon_{ab} [\Delta_{-}^a, [\Delta_{-}^b, \Phi]] e^{\frac{i}{\hbar}X} \\ &= \left(\frac{\hbar}{i}\right) \varepsilon_{ab} \int d\mu \left[(\Delta_{+}^a e^{\frac{i}{\hbar}W}) ([\Delta_{-}^b, \Phi] e^{\frac{i}{\hbar}X}) + [\Delta_{-}^b, \Phi] \Delta_{-}^a e^{\frac{i}{\hbar}X} \right] \\ &= 0, \end{aligned} \quad (30)$$

as follows by using the Master Equations (27). The gauge freedom is parametrized in terms of one bosonic function Φ .

As in the case without $Sp(2)$ symmetry [1], we can also here understand the independence of the gauge-fixing function X from the existence of a BRST symmetry. Introduce BRST operators σ_W^a and σ_X^a by [8]

$$\begin{aligned} \sigma_W^a F &\equiv \frac{\hbar}{i} e^{-\frac{i}{\hbar}W} [\Delta_{+}^a, F] e^{\frac{i}{\hbar}W}, \\ \sigma_X^a F &\equiv \frac{\hbar}{i} e^{-\frac{i}{\hbar}X} [\Delta_{-}^a, F] e^{\frac{i}{\hbar}X}. \end{aligned} \quad (31)$$

These generalize in an obvious way the $Sp(2)$ -covariant “quantum BRST operator”[§] derived in ref. [17] to the case of a higher-order formalism. The Master Equations for W and X are preserved under transformations $W \rightarrow W + \varepsilon_{ab} \sigma_W^a \sigma_W^b F$ and $X \rightarrow X + \varepsilon_{ab} \sigma_X^a \sigma_X^b F$, respectively. Thus, the statement of gauge independence of the path integral (28) can, equivalently, be rephrased as $Z_X = Z_{X+\delta X}$ with $\delta X = \varepsilon_{ab} \sigma_X^a \sigma_X^b F$. The formalism is symmetric under exchanges of X with W ; only boundary conditions stipulate which part will play the rôle of gauge fixing (here chosen to be X).

It is straightforward to derive, using eq. (26), the expected properties of the above BRST operators. For example, if we define BRST invariant operators G by $\sigma_W^a G = 0$, then their expectation values will not depend on X :

$$\langle G \rangle_{X+\delta X} - \langle G \rangle_X = 0, \quad (32)$$

and we similarly find $\langle \sigma_W^a F \rangle = 0$ for any F . Note that, as expected, the σ^a 's satisfy the $Sp(2)$ -algebra $\sigma^{\{a} \sigma^{b\}} = 0$ [8].

Let us now finally consider the analogue of second-class constraints in this higher-order triplectic formalism. We can proceed essentially as in the previous case, and shall therefore be brief. In the second-order formalism, one can define the matrix of second-class constraints Θ^a by [14]

$$E^{\alpha\beta a} \equiv (\Theta^a, \Theta^b)^a, \quad (33)$$

where an inverse matrix, in the following sense, is assumed to exist:

$$E^{\alpha\beta a} Y_{ab\beta\gamma}^c = \delta_b^c \delta_{\gamma}^a. \quad (34)$$

A direct triplectic counterpart of the Dirac antibracket is then [14]

$$(A, B)_D^a \equiv (A, B)^a - (A, \Theta^b)^b Y_{bc\beta\gamma}^a (\Theta^c, B)^c, \quad (35)$$

[§]For the case without $Sp(2)$ symmetry, see ref. [16].

and similarly one can introduce a nilpotent Dirac $\Delta_D^{(2)a}$ -operator [14]. In the higher order formalism, apply the deformation theorem [10] as before, and introduce, by means of a transformation $U(\Gamma, \partial)$,

$$\begin{aligned}\Delta_{D\pm}^a &= U^{-1}\Delta_{D\pm}^{(2)a}U = \Delta_{D\pm}^a + \mathcal{O}(\hbar) \\ \Theta^\alpha &= U^{-1}\Theta^{(2)\alpha}U = \Theta^{(2)\alpha} + \mathcal{O}(\hbar) .\end{aligned}\quad (36)$$

Again, by construction,

$$[\Delta_{D\pm}, \Theta^\alpha] = 0 \quad , \quad [\Theta^\alpha, \Theta^\beta] = 0 . \quad (37)$$

Solutions of the Master Equations (27), with Δ_\pm^a replaced by $\Delta_{D\pm}^a$, can then again be expressed in terms of the solutions to the second-order equations:

$$e^{\frac{i}{\hbar}W} = U^{-1}e^{\frac{i}{\hbar}W^{(2)}} \quad , \quad e^{\frac{i}{\hbar}X} = U^{-1}e^{\frac{i}{\hbar}X^{(2)}} . \quad (38)$$

They can also be expanded in terms of $Sp(2)$ -covariant higher antibrackets (see ref. [8]) $\Phi_{\Delta_\pm^a}^{(k)a}$ which satisfy an $Sp(2)$ generalization of the usual strongly homotopy Lie algebra. Using the definition in terms of nested commutators acting on unity [8], also these higher antibrackets can be taken with operators as entries. For physical (in the sense given above) and mutually commuting operators, we find the expected generalization:

$$\Phi_{\Delta_{D\pm}}^{(k)a}(A_1, \dots, A_{i-1}, \Theta^\alpha, A_{i+1}, \dots, A_k) = 0 . \quad (39)$$

The triplectic second-class constraints reduce the dimension of the “triplectic manifold” to $6(N - K)$. The obvious $Sp(2)$ -generalization of the Lagrangian path integral in the presence of these constraints is precisely as in (22). The only difference is that W and X satisfy the appropriate $Sp(2)$ Master Equations (27). It is straightforward to verify the gauge independence of this path integral, exactly as in eq. (30).

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